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Random Walk Statistics on Fractal Structures

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We consider some statistical properties of simple random walks on fractal structures viewed as networks of sites and bonds: range, renewal theory, mean first passage time, etc. Asymptotic behaviors are shown to be controlled by the fractal ($\bar{d}$) and spectral ($\tilde{d}$) dimensionalities of the considered structure. A simple decimation procedure giving the value of $\tilde{d}$ is outlined and illustrated in the case of the Sierpinski gaskets. Recent results for the trapping problem, the self-avoiding walk, and the true-self-avoiding walk are briefly reviewed. New numerical results for diffusion on percolation clusters are also presented.

KEY WORDS: Random walk statistics; fractal structures; spectral dimension; percolation clusters.

1. INTRODUCTION

During the last year, the family of fractal structures encountered in condensed matter physics has been considerably enriched. After the percolation clusters, which have been considered as the major example, a lot of fractal objects have been discovered. We quote only two examples: (i) fractal aggregates[1] obtained from diffusion-limited growth processes, and (ii) a remarkable variety of adsorbents[2] (fractal surfaces). The fractal dimension $\bar{d}$ emerges as a first operative measure of the fractal geometry. However it is now recognized that fractal structures require the definition of (at least) three dimensions: d, the dimension of the embedding Euclidean space; $\bar{d}$, the fractal dimension; $\tilde{d}$ the spectral dimension.[3] For Euclidean spaces, these three dimensions are equal. This can be construed as an "accidental" degeneracy. The spectral dimension $\tilde{d}$ is naturally associated with the power law of the low-frequency density of states (e.g., for elastic vibrations) $\rho(\omega) \sim \omega^{\tilde{d}-1}$. Recently, it has been shown[4-6] that simple physical problems on fractals (classical diffusion, quantum localization, etc.)

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are governed by this spectral dimension. In this review, we address the first problem of random walk (RW) on fractal lattices. The fractal structures considered are viewed as networks of sites and bonds (graphs). In Section 2, various statistical properties of RWs are reviewed. Section 3 is mainly devoted to two illustrative examples: trapping of the RW and the diffusion on percolation clusters. In Section 4, we shall summarize some recent results relating to self-avoiding walks (SAW) abd true self-avoiding walks (TSAW) on fractal lattices. In the concluding section we will give a nonexhaustive list of open questions, which call for further investigation in the future.

2. RANDOM WALK STATISTICS

Classical diffusion and random walks in Euclidean spaces are a well-known field of physics.[8] Several properties of related interest will be investigated in general: spatial distribution, range, renewal theory, etc. For instance, the mean-squared displacement from the origin at time t (or after N steps on a lattice), behaves asymptotically as[4,7]: $\langle \mathscr{R}^2 \rangle \sim N^{\tilde{d}/\bar{d}}$, i.e., $\nu_{\mathrm{RW}} = \tilde{d}/2\bar{d}$. As it should be, the standard value $\nu_{\mathrm{RW}} = 1/2$ is recovered on an Euclidean lattice, where $\tilde{d} = \bar{d}$. The exponent ν_{RW} appears here as a combination of the fractal ($\bar{d}$) and spectral ($\tilde{d}$) dimensions. This combination enters into the mean-squared displacement because the measure of Euclidean distances brings with it the fractal dimension. In order to obtain "pure" quantities, where the spectral dimension enters alone, one has to consider other RW properties which are reviewed below.

2.1. Main Result

2.1.1. Range of the RW. The range of a simple RW, R_N is defined as the number of distinct sites visited up to step N. The behavior of the expectation value $S_N = \langle R_N \rangle$ and the variance $\sigma_N^2 = \langle R_N^2 \rangle - \langle R_N \rangle^2$ is well known on Euclidean lattices[8] of dimension d, and which is of interest for various problems. For instance ($N \gg 1$),

$$S_N \simeq (8/\pi)^{1/2} N^{1/2}, \qquad \sigma_N^2 \simeq N \text{ at } d=1$$
$$S_N \simeq N/\ln N, \qquad \sigma_N^2 \simeq N^2/\ln^4 N \text{ at } d=2 \tag{1}$$

and $S_N \simeq N$, $\sigma_N^2 \simeq N \ln N$ at $d = 3$.

On a fractal structure, the behavior of R_N is controlled by the spectral dimension $\tilde{d}$ only. The following properties hold in general on fractal structures.[6,9]

(i) The asymptotic behavior ($N \gg 1$) of S_N is given by

$$S_N \sim N^{\tilde{d}/2} \text{ if } \tilde{d} < 2 \text{ and } S_N \sim N \text{ if } \tilde{d} \geqslant 2 \tag{2}$$

In particular, $1 - F = \lim_{N \to \infty} S_N/N = 0$ at $\tilde{d} < 2$.

(ii) For structures with $\tilde{d} < 2$, σ_N^2 behaves as $\sigma_N^2 \simeq N^{\tilde{d}}$ and, the ratio $\rho_N = \sigma_N/S_N$ converges toward a finite value ρ_∞ at $N = \infty$, which is a decreasing function of $\tilde{d}$. For structure with $\tilde{d} \geqslant 2$, the limiting value ρ_∞ is zero.

(iii) The reduced random variable $X = R_N/S_N$ converges in distribution to a proper law for $\tilde{d} < 2$ and to a degenerate law ($X \to 1$ with probability one) for $\tilde{d} > 2$. The presence of a distribution for the ratio X reveals the strong fluctuations of the range R_N for $N \gg 1$. $\tilde{d} = 2$ appears in this sense as an upper critical dimension for the RW problem on fractals.

The numerical results[9] obtained on fractals are in good agreement with the above properties. For instance, on a two-dimensional Sierpinski gasket, the obtained exponent for S_N as given by Eq. (2) is 0.682 ± 0.005 to be compared with the exact value $\tilde{d}/2 = 0.68260$.

2.1.2. Renewal Theory. Another measurement of $\tilde{d}$ is provided by the renewal theory, describing mainly the returns of the walker to the original site. Let $P_0(N)$ be the probability of return after N steps, ν_N the number of returns during that time and τ_n the time of the nth return to the origin. The asymptotic behavior of these related quantities can be summarized as follows.

(i) The probability $P_0(N)$ of closed walks of length N is given by[4] $P_0(N) \sim N^{-\tilde{d}/2}$ for all $\tilde{d}$. This result can be viewed as the generalization of the corresponding result on Euclidean lattices: $P_0(N) \sim N^{-d/2}$. More generally, it is possible to define a set of critical exponents, in anology with the Gaussian model of phase transitions. It is interesting to note that, on a fractal structure, it is the fractal dimensionality $\bar{d}$ which replaces d in the Josephson scaling law: $d\nu = 2 - \alpha$. In general[10] the combination $\bar{d}\nu$ must be independent of $\bar{d}$.

(ii) The time τ_n until the nth return, when properly normalized ($u = \tau_n/n^{1/\alpha}$) admits an asymptotically stable distribution $p(u; \alpha, 1)$ with $\alpha = 1 - \tilde{d}/2$ at $\tilde{d} < 2$, $\alpha = 0$ at $\tilde{d} > 2$. In particular, $\Pr(\tau_n/n^{1/\alpha} \geqslant u) \simeq u^{-\alpha}$ at $u \gg 1$ and $n \gg 1$.

(iii) The related number ν_N giving the number of returns behaves as a power law: $\nu_n \simeq N^\alpha$, and $\Pr(\nu_N/N^\alpha \leqslant x) \simeq 0(1)$ at $x \gg 1$ and $N \gg 1$.

From the statement (ii), we deduce: $\lim_{N \to \infty} \tau_N/N = \infty$ at $\tilde{d} < 2$. This means that if τ denotes the time of the first return to the original site, then τ

is finite with probability one, whereas its mean value is infinite. Incidentially, it appears from (iii) that for $N \to \infty$, the number of returns to the origin in N steps increases proportionally to N^{α} and not to N, as might be expected from the recurrence property of the RW at $\tilde{d} < 2$.

2.1.3. Spatial Distribution. In general the spatial distribution of a RW is another interesting property, providing some additional information relating to the exploration of space. Using the standard Euclidean metric, the normalized spatial distribution of the RW position $\mathbf{r}$, starting from a chosen site $\mathbf{r} = 0$ can be written $P_N(\mathbf{r}) = R_0^{-\bar{d}} f(r/R_0)$ where $r = |\mathbf{r}|$ denotes the distance from the origin at step N, R_0 is the "radial" extension of the walk $R_0 \sim N^{\nu_{\mathrm{RW}}}$ where $\nu_{\mathrm{RW}} = \tilde{d}/2\bar{d}$. The function $f(u)$ of the reduced variable $u = r/R_0$ is not simple in general, in contrast with the Euclidean lattices where $f(u \gg 1) \sim \exp(-u^{\delta})$, $\delta = (1 - \nu_{\mathrm{RW}})^{-1}$. However, the general form of $P_N(\mathbf{r})$ permits one to extract some interesting information. For instance, $\Pr(r > \xi) \sim \xi^{-1/\nu_{\mathrm{RW}}}$ as $\xi \to \infty$. The cumulative time spent at the origin between step 0 and step N is given by $\int_0^N P_N(r = 0)\, dN \sim N^{\alpha}$ as expected. Accordingly the fractal dimensionality D of the zero-crossing set is given by $D = 1 - \tilde{d}/2$ if $\tilde{d} < 2$ and $D = 0$ if $\tilde{d} > 2$. Another interesting quantity, which can be extracted from $P_N(r)$ is given by the mean first passage time $T_1(\xi)$ at a distance ξ from the origin. It can be shown[9] that $T_1(\xi)$ is given by

$$T_1(\xi) = \int_0^{\infty} dN \int_0^{\xi} r^{\bar{d}-1} P_N(\mathbf{r})\, dr \simeq \xi^{1/\nu_{\mathrm{RW}}} \qquad \text{at} \quad \xi \to \infty$$

providing a useful method for the calculation of the exponent ν_{RW}. The special role played by $\tilde{d} = 2$ in the above discussion, appears also in the spatial behavior of the RW. Lets define $\Sigma_N \simeq R_0^{\bar{d}}$ to be the number of effectively accessible sites during N steps. It is clear that $\Sigma_N \simeq S_N$ for $\tilde{d} \leqslant 2$ (compact exploration) and $\Sigma_N \gg S_N$ at $\tilde{d} > 2$ (noncompact exploration).

2.2. Decimation Procedure

Because of dilation invariance, fractals lend themselves particularly conveniently to scaling approaches. In the following we shall illustrate such a scaling calculation for the RW properties on the Sierpinski gaskets. In that case[5] $\bar{d} = \ln(d + 1)/\ln 2$ and $\tilde{d} = 2 \ln(d + 1)/\ln(d + 3)$.

In general, if $P_n(\mathbf{r})$ denotes the probability that a RW is at site $\mathbf{r}$ after n steps, we have the recurrence

$$P_{n+1}(\mathbf{r}) = \sum_{\boldsymbol{\delta}} p(\boldsymbol{\delta})\, P_n(\mathbf{r} - \boldsymbol{\delta})$$

where $p(\boldsymbol{\delta})$ denotes the transition probability for a displacement vector $\boldsymbol{\delta}$. Almost all information relevant to the RW statistics is contained in the generating function $G(\mathbf{r}, z)$ defined for complex z by $G(r, z) = \sum_n z^n P_n(r)$ and given by the solution of the following equation:

$$G(\mathbf{r}, z) - z \sum_{\boldsymbol{\delta}} p(\boldsymbol{\delta})\, G(\mathbf{r} - \boldsymbol{\delta}, z) = \delta_{\mathbf{r},0}$$

For instance, $G(\mathbf{r} = 0, z)$ gives directly the probability of return to the original site after n steps. The generating function $S(z)$ for the average number S_N of visited sites is also given by[8]

$$S(z) = [(1 - z)^2\, G(0, z)]^{-1}$$

Moreover, the corresponding generating function for the number of returns is simply given by $(1 - z)^{-1}\, G(0, z)$. These general results are valid on Euclidean spaces as well as on fractals. In general, $G(0, z)$ exhibits a singular behavior at $z = 1$, following the generic form: $G(0, z) \sim (1 - z)^{-\alpha}$, $z \lesssim 1$. This singularity governs the asymptotic behavior of $P_n(0)$, S_n, etc. at $n \gg 1$. The value of the exponent α is therefore closely related to the spectral dimension $\tilde{d}$. Its precise value can be extracted using the following decimation procedure.

Let consider the example of the Sierpinski gasket in d dimensions, where $p(\boldsymbol{\delta}) = 1/2d$. Starting from the equations of $G(\mathbf{r}, z)$ written for all $\mathbf{r}$, one eliminates the amplitudes corresponding to the sites located at midpoints of hypertetrahedron edges at the lowest scale. This decimation procedure leads to a reduced set of equations describing the same physics, on a gasket, scaled down by a factor $b = 2$. This exact renormalization leads to a renormalized z and renormalized $G(0, z)$. For instance, at $d = 1$ (linear chain) the renormalization equation can be written

$$z' = z^2/(2 - z^2)$$

and

$$G(0, z') = K_1(z)\, G(0, z), \qquad K_1(z) = (1 - z^2/2)$$

Close to the unstable fixed point $z^* = 1$, $K_1(z) \sim 1/2$, $(1 - z') \sim 4(1 - z)$, giving the expected known value of α: $\alpha = \ln 2/\ln 4 = 1/2$.

For $d = 2$, the same procedure yields similar expressions:

$$z' = z^2/(4 - 3z)$$

and

$$G(0, z') = K_2(z)\, G(0, z), \qquad K_2(z) = (2 + z)(4 - 3z)/(4 + z)(2 - z)$$

In the vicinity of $z^* = 1$, we have $K_2(z) \sim 3/5$ and $(z' - 1) \sim 5(z - 1)$ and $\alpha = \ln(5/3)/\ln 5$ as expected.

More generally, in dimension d, one obtains the expected value of the exponent α: $\alpha = 1 - \tilde{d}/2$, where $\tilde{d} = 2 \ln(d + 1)/\ln(d + 3)$. As can be seen the function $G(0, z)$ is in general the solution of an interesting nontrivial functional equation. However, the leading exponent α which governs the singular behavior at $z \sim 1$ is simply given by the associated linearized equation. The above-discussed procedure is very simple and may be useful in the investigation of more complicated related problems. Similar procedures were used previously[6,11] in the investigation of other linear problems, like the spectrum of the Schrödinger equation or the spectrum of harmonic excitation on the gaskets.

3. TWO ILLUSTRATIVE EXAMPLES

In this section we will illustrate the above results on two examples. The first is the calculation of the survival probability $\Phi(t)$ of the RW in the presence of perfect traps. The second is the known example of the percolation clusters at threshold. Other examples are briefly discussed in the next section.

3.1. Fractal Structure with Traps

The excitation dynamics in the presence of a random distribution of trapping centers is known to be an important topic in condensed matter physics with application to optical spectroscopy, nuclear magnetic resonance, and particle transport. The motion of a particle (e.g., exciton) performing a RW on a regular lattice containing a given distribution of traps (binary systems: traps and active sites) provides a relatively good picture. Ternary crystals with three components (traps, active and binary molecules) lead to problems of a RW on a dilute lattice, in presence of traps. Keeping in mind the percolation clusters as a main example of practical interest, it is interesting to investigate the RW trapping on a general fractal structure.[12,13] As can be shown below, all the decay properties are governed by the spectral dimension $\tilde{d}$.

Assume that the traps are distributed randomly on a fractal lattice, and they occupy the lattice sites with probability p. Traps are assumed to be perfect: the RW is instantaneously absorbed at the first encounter of a trap. The survival probability $\Phi(t)$ at time t can be expressed in terms of the RW properties on a trap-free lattice

$$\Phi(t) = e^{\lambda} \langle \exp(-\lambda R_t) \rangle$$

where R_t denotes the range of the RW (see Section 2.1) on a trap-free lattice and $\lambda = -\ln(1-p)$. The origin of the walk is assumed to be not a trap and the average is taken over all possible realizations of the RW in space and time.

3.1.1. Various Approximations. The above expression for $\Phi(t)$ shows that the survival probability is completely dominated by the statistical properties of the range R_t and therefore by the spectral dimension. Various approximations must therefore show this property. For instance, the first-order cumulant expansion of $\Phi(t)$, giving the short-time behavior, leads to $\Phi(t) \sim \exp(-C \cdot \lambda t^{\tilde{d}/2})$ at $\tilde{d} < 2$ and $\exp(-C' \cdot \lambda t)$ at $\tilde{d} > 2$. Here C and C' denote numerical factors. This result is valid at short time and all concentrations of traps. In order to go beyond the first-order approximation, we need the knowledge of the distribution of R_t in great detail. However, more can be obtained in the limit of low concentration. As a first approximation, we can introduce the average time of capture T^*, define by $p \cdot S_T^* \simeq 1$, which leads to

$$T^* \sim p^{-2/\tilde{d}} \text{ at } \tilde{d} < 2 \text{ and } p^{-1} \text{ at } \tilde{d} \geqslant 2$$

In the spirit of an effective medium theory, the survival probability assumes an exponential decay, given by $\Phi(t) \simeq \exp(-t/T^*)$. This approximation, which is equivalent to the first term of the expansion in powers of the concentration p is valid at $t \gg T^*$. At short time ($t \ll T^*$), the previous expression must be replaced by $\Phi(t) \simeq 1 - S_t/S_{T^*}$.

3.1.2. Fluctuation Mechanism. The above approximation does not take into account the fluctuations of the trap density. It is clear however, that particle survival for long periods will occur only in sufficiently large trap-free regions. These regions are rare, but will govern the limit of large t. A standard argument leads to the following result:

$$\Phi(t) \simeq \exp(-a \cdot p^y t^x), \qquad x = \tilde{d}/(\tilde{d}+2) \text{ and } y = 1 - x = 2/(\tilde{d}+2)$$

As expected, the fractal dimension $\bar{d}$ is absent in the final expression of $\Phi(t)$. This result generalizes the known $d/(d+2)$ law on Euclidean spaces,[14] obtained some years ago. Note that the argument leading to $\tilde{d}/(\tilde{d}+2)$ law is similar to that giving the Lifshitz singularity. It can be shown that on a fractal lattice, such a singularity (if any) is controlled by $\tilde{d}$: we have simply to replace d by $\tilde{d}$.

The range of validity of the fluctuation mechanism can be found by comparing the following two length scales: the diffusion length ξ_D and the average distance $l \sim p^{-1/\bar{d}}$ between two traps. Let us denote by $t^* \sim p^{-2/\tilde{d}}$ the crossover time where $l \sim \xi_D$. The temporal behavior of $\Phi(t)$ is then governed by two time scales: T^* and t^*. Following the previous analysis, two cases are to be distinguished.

(i) $\tilde{d} < 2$. In that case, the RW is recurrent and $t^* \sim T^* \sim p^{-2/\tilde{d}}$: only one time scale is involved and

$$\Phi(t) \simeq \begin{cases} 1 - C \cdot pt^{\tilde{d}/2} & \text{at} \quad t \ll t^* \\ \exp(-(t/t^*)^x) & \text{at} \quad t \gg t^* \end{cases}$$

(ii) $\tilde{d} > 2$. For such a transient RW the effective medium theory remains valid up to times such that $t/T^* \ll (t/t^*)^x$, i.e., $t \ll p^{-\tilde{d}/2}$. Therefore ($t^* \ll p^{-d/2}$) one obtains

$$\Phi(t) \simeq \begin{cases} \exp(-t/T^*) & \text{at} \quad T^* \ll t \ll p^{-\tilde{d}/2} \\ \exp(-(t/t^*)^x) & \text{at} \quad t \gg p^{-\tilde{d}/2} \end{cases}$$

At short time ($t \ll T^*$), the first expression hold also because $T^* \sim p^{-1}$. Moreover, for vanishing p, $\Phi(t) \simeq \exp[-(1-F)t \cdot p]$ as expected, where F has been defined in Section 2.1.

3.1.3. Scaling Behavior at $\tilde{d} < 2$. Let consider the interesting case $\tilde{d} < 2$, which corresponds to a compact exploration of space. The expression of $\Phi(t)$ can be viewed as a function of the reduced random variable $X = R_t/S_t$ defined in Section 2.1:

$$\Phi(t) = e^{\lambda} \langle \exp(-\lambda t^{\tilde{d}/2} \cdot X) \rangle$$

Using the result (Section 2.1) relative to the probability distribution of the random variable X, we deduce the following scaling expression for $\Phi(t)$, which is valid at all times and all concentrations:

$$\Phi(t) = \exp(\lambda - \varphi(\lambda t^{\tilde{d}/2}))$$

Here $\varphi(v)$ denotes a universal function of the scaling variable $v = \lambda t^{d/2}$. In the limit of low concentration ($\lambda \sim p$), we recover the previous results with $\varphi(v) \simeq v^{2/(2+\tilde{d})}$ at $v \gg 1$ and, $\varphi(v) \sim v$ at $v \ll 1$.

3.2. RW on Percolation Clusters

At a percolation threshold, the infinite percolation cluster is a fractal object.[15] Therefore, the concepts of the previous sections can be brought to bear on the physics in the vicinity of the percolation threshold. It was recognized early that the fractal dimension of a percolation cluster at threshold can be expressed in terms of the critical exponents of the percolation transition ($\beta_p, \nu_p,$... indexed by p for percolation): $\bar{d} = \delta_p \beta_p / \nu_p$. Using the hyperscaling relation, $\bar{d}$ reduces to the familiar expression: $\bar{d} = d - \beta_p/\nu_p$. Note that $\bar{d} = 4$ for $d \geqslant 6$ in contrast to the last expression of $\bar{d}$. This behavior of $\bar{d}$ may be traced to the breakdown of hyperscaling above the critical dimension.

The spectral dimensionality $\tilde{d}$ of the percolation cluster can be derived via a simple crossover argument. The result[4,5] is $\tilde{d} = 2(d\nu_p - \beta_p)/(t - \beta_p + 2\nu_p)$, where t denotes the conductivity exponent. This result is valid only at $d \leqslant 6$, and $\tilde{d}$ sticks at 4/3 for $d \geqslant 6$. For instance, on the Bethe lattice ($d = \infty$) we have[16] $\bar{d} = 4$ and $\tilde{d} = 4/3$, which are the mean field values for the percolation problem. Close to but above the percolation threshold ($p \gtrsim p_c$), the infinite cluster retains its local self-similarity up to the correlation length $\xi_p \sim |p - p_c|^{-\nu_p}$. Fractal to Euclidean crossovers are then expected to occur, and were studied by various authors.[7,16] In addition to the whole percolation cluster, its biconnected component is also a fractal object[17]: the values of $\bar{d}$ and $\tilde{d}$ are given by the same expressions above, where β_p is replaced by β_B (ν_p and t take the same values). The physics on the percolation cluster is closely related to that on its backbone. In fact, the localization exponent[5] $\beta_L = (\bar{d}/\tilde{d})(\tilde{d} - 2)$ takes the same value ($=d - 2 - t/\nu_p$) on these two structures.[17] Taken into account the previous results, the average of various physical quantities over the size cluster distribution, as well as finite size effects are very easy to understand[16,17] and will not be discussed here.

3.2.1. Universality of the Spectral Dimension: $\tilde{d} = 4/3$. Empirically it was found[4] that the spectral dimension $\tilde{d}$ as determined by the previous expression, with known estimates of t, ν_p and β_p, apears to be numerically close to 4/3, for all dimensions $2 \leqslant d \leqslant 6$. This finding leads to the remarkably simple prediction[5] for S_N:

$$S_N \simeq N^{2/3} \qquad \text{on percolating clusters}$$

Actually, an argument has been presented,[5,18] which suggests that the 2/3 might be an exact and not only approximate value. Define the "open frontier" as the number of fresh sites adjacent to the visited sites during an N-step random walk. It is given by $F_N = S_N dS_N/dN$ and behaves asymptotically as: 0 for $p < p_c$ and S_N for $p > p_c$. If one assumes that, at the threshold $p = p_c$, the open frontier is marginally equal to the Gaussian fluctuation in the number of accessible sites, owing to the random percolation process, then $F_N \sim S_N^{1/2}$ and the 2/3 law follows.

3.2.2. Numerical Results for $\tilde{d}$. The 2/3 law has also been checked by computer experiment in two and three dimensions, using direct[9] or indirect[19] estimations of $\tilde{d}$. Increasing confidence in this 2/3 law comes from recent direct[20] numerical determination of the exponent $\tilde{d}/2$, using the law $S_N \sim N^{\tilde{d}/2}$ at threshold for all dimensions $2 \leqslant d \leqslant 6$ and $d = \infty$. The cluster sizes M were chosen to be much larger than S_N so as to avoid finite size effects ($S_N/M \ll 1$). Averages were taken over 10^4 to 2×10^4 runs of N-step walks (up to $N \simeq 4 \times 10^4$) of randomly chosen origin on the cluster. The size of the simulated cluster was much greater than 10^4, for all dimensions. More extensive calculations were performed at $d = 2$ up to $N \simeq 10^6$ on clusters containing much more than 10^6 sites and inscribed inside boxes of size 1280×1280. The corresponding boxes at $d = 3$ had sides of order 300. From the numerical data, three points are to be noted: (i) S_N (vs. N) takes very similar values for all dimensions, including $d = \infty$; (ii) the asymptotical slope of S_N is the same for all d, up to an accuracy of order 5×10^{-3}; (iii) a systematic curvature is observed at all d, before reaching the asymptotic limit. The observed deviations follow the same pattern for all d, including the Bethe lattice case ($d = \infty$). The corrections to scaling are therefore very important in the determination of the exponent of S_N. The appropriate expression for S_N takes the following form:

$$S_N = aN^s \left(1 + \frac{b}{a} N^{-\omega} + \cdots\right) \tag{3}$$

where $s = \tilde{d}/2$ and ω is the correction to scaling exponent. Trial values for ω and s were used in order to extract their precise values. The analysis of the numerical data shows that $s(d = 6) - s(d)$ is smaller than 5×10^{-3} for all $d \leqslant 6$. More surprising is the "universal" value of the exponent ω: $\omega \simeq 1/6$ for all $d \geqslant 2$, including $d = \infty$. Including exaggerated error bars leads to the finding: $1/6 \lesssim \omega < 1/4$ for all d. In Table I, are shown the values of coefficients a and b in Eq. (3) as obtained from the numerical data.

The results support strongly the prediction of the open frontier argument, with very good accuracy. On the other hand, the presence of a "Gaussian" correction $N^{s-\omega} \sim N^{1/2}$ supports *a posteriori* the basic picture of this argument. Note that at $d = 1$, $S_N \sim N^{1/2}$ with a correction of order $N^{-1/2}$. More generally the correction term to S_N on a regular fractal can be shown to behave like $N^{\tilde{d}/2-1}$. Therefore, the observed correction $N^{1/2}$ is particular and specific to the percolation clusters. The second comment is about the prediction of the $\varepsilon = 6 - d$ expansion. Using the published[21] results for different exponents (ν_p, β_p, and t), one obtains: $d = 4/3 + k\varepsilon^2$, where $k = 761/83349 \simeq 9 \times 10^{-3}$ is a very small positive number. At least for $d = 5$, i.e., $\varepsilon = 1$, where the ε expansion is expected to provide a reliable estimation, there is no contradiction with the numerical estimation (up to the numerical accuracy). However, a further reexamination[21] of the ε

Table 1. Numerical Values of the Coefficients a and b[a]

d	p_c	a	b
2	0.593	0.93	0.60
3	0.312	0.84	0.56
4	0.197	0.78	0.58
5	0.141	0.75	0.60
6	0.108	0.72	0.58
∞	0.500	0.55	0.74

[a] See Eq. (3) for the expression of S_N. $d \equiv$ Euclidean dimension, $p_c \equiv$ site percolation threshold ($d = \infty$ refers to the Bethe lattice of coordination number $z = 3$).

expansion would be welcome in order to resolve the discrepancy at lower dimensions. We believe that the 2/3 law is actually a very good prediction, which calls for a rigorous proof or disproof. Flory-like arguments in polymer physics provide another example of inexact but very good predictions.

4. CORRELATED RW ON FRACTALS: SAW AND TSAW

In the preceding section, it was shown that RWs have simple properties and provide a powerful probe giving access to the spectral dimension $\tilde{d}$. The next step is naturally the study of the self-avoiding walks (SAW) and the true self-avoiding walks (TSAW) on fractal lattices. The first motivation is that different walkers explore differently and tell complementary stories. The second motivation is the remarkable success of the Flory approximation for SAW and TSAW on Euclidean spaces: is it accidental or not? In the following we shall summarize the main results on the subject.

4.1. Self-Avoiding Walks

Since only the end parts of SAWs can lie on dead ends, the asymptotic behavior of their radius is expected to be dominated by the structure of the backbone (i.e., doubly connected component) rather than by that of the full fractal space. The first important result[10,22] is that the combination $\bar{d}\nu$ is an intrinsic property, independent of the space in which the fractal is embedded, whereas $\bar{d}$ and ν depend both on this embedding. Here $\bar{d}$ refers to the fractal dimension of the backbone. This observation leads, under the assumption that only $\tilde{d}$ plays a role, to the following Flory formula,[22] for fractals

$$\nu_F = \frac{\tilde{d}}{\bar{d}} \frac{3}{\tilde{d}+2}, \qquad \tilde{d} \leqslant 4$$

This is the simplest approximation that reduces to the Flory form for Euclidean spaces ($\tilde{d}$ refers to the backbone). The value $\tilde{d} = 4$ appears also as the upper critical dimension for the SAW problem on fractals. A direct argument[5] showing that excluded-volume effects are negligible at $\tilde{d} > 4$ can be outlined using the fractal dimension of the RW.

Exact results for various lattices[22] show that this approximation is not very satisfactory and that properties of SAW depend on other intrinsic aspects of the fractal. For instance, for the Sierpinski gasket, we found $\nu = 0.798$ in two dimensions and $\nu = 0.729$ in three dimensions, to be compared with the above prediction for ν_F. This suggests that the success of the Flory formula for Euclidean lattices is somewhat accidental, and that for general spaces there exists no comparable formula combining simplicity and accuracy. The implications of the above results for the controversial problem of SAW on percolating clusters will be found elsewhere.[22]

4.2. True Self-Avoiding Walks[23]

Under the assumption that only $\tilde{d}$ plays a role, the same argument presented above suggests a similar approximation for the TSAW on fractals

$$\nu = \frac{\tilde{d}}{\bar{d}} \frac{2}{\tilde{d}+2}, \qquad \tilde{d} \leqslant 2$$

This approximation reduces to the corresponding expression for Euclidean spaces,[24] and shows that $\tilde{d} = 2$ plays the role of the upper critical dimension of TSAWs. Numerical simulations[25] on the two-dimensional Sierpinski gasket gives a surprisingly close value for the exponent ν: 0.510 ± 0.005 to be compared with the above prediction: 0.5119. On the gasket, the exploration of space remains compact in spite of the repulsion. Given a compact volume which contains the walker, most points inside this volume are visited before a new site outside the volume is explored. An example of compact exploration is of course the one-dimensional RW or TSAW.[25]

In view of the validity of the relations giving ν for SAW and TSAW, it is of great interest to check this working hypothesis on different fractal structures. A more detailed understanding of the universality classes for SAW and TSAWs on fractals will be necessary to resolve that question.

5. CONCLUSION

To conclude, we shall list some questions, which call for investigation in the near future.

1. Is $\tilde{d} = 4/3$ true or only a very good approximation on percolation clusters?

2. What are the precise values of $\tilde{d}$ on other fractal structures, like the diffusion-limited aggregates[27]?

3. Are there other dimensions, such as the connectivity dimension, which play a key role in the understanding of SAWs problems on fractals?

The investigation of different variants of RWs on fractal lattices would be welcome, in view of a more detailed unerstanding of the universality classes for RWs on fractal lattices.

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